

Onset of superradiant instabilities in the hydrodynamic vortex model

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(Dated: August 1, 2014)

The hydrodynamic vortex, an effective spacetime geometry for propagating sound waves, is studied *analytically*. In contrast with the familiar Kerr black-hole spacetime, the hydrodynamic vortex model is described by an effective acoustic geometry which has no horizons. However, this acoustic spacetime possesses an ergoregion, a property which it shares with the rotating Kerr spacetime. It has recently been shown numerically that this physical system is linearly unstable due to the superradiant scattering of sound waves in the ergoregion of the effective spacetime. In the present study we use analytical tools in order to explore the onset of these superradiant instabilities which characterize the effective spacetime geometry. In particular, we derive a simple analytical formula which describes the physical properties of the hydrodynamic vortex system in its critical (marginally-stable) state, the state which marks the boundary between stable and unstable fluid configurations. The analytically derived formula is shown to agree with the recently published numerical data for the hydrodynamic vortex system.

I. INTRODUCTION

One of the most remarkable characteristics of the rotating Kerr black-hole spacetime [1] is the existence of an *ergoregion* [2]: a region in which all physical objects must co-rotate with the spinning black hole. In particular, no physical observer inside the ergoregion can remain at rest with respect to inertial asymptotic observers.

The presence of the ergoregion in the Kerr black-hole spacetime is responsible for the intriguing phenomenon of *superradiant* wave scattering: it was first realized by Zel'dovich [3] (see also [4, 5]) that a co-rotating bosonic field of the form $e^{im\phi}e^{-i\omega t}$ interacting with a spinning Kerr black hole of angular velocity Ω can be amplified (gain energy) if the incident wave field satisfies the superradiant condition

$$\omega < m\Omega . \quad (1)$$

The superradiant scattering of bosonic fields in the Kerr black-hole ergoregion (the extraction of rotational energy from the black hole) has the potential to destabilize the spacetime geometry [6]. However, the Kerr black hole is known to be stable against massless perturbation fields [5, 7]. The stability of the Kerr spacetime against the superradiant scattering of massless bosonic fields in its ergoregion may be attributed to the absorption properties of the black-hole horizon [6]. In particular, the black-hole horizon acts as a one-way membrane which absorbs the (potentially dangerous) perturbation fields before any instability has the chance to develop in the ergoregion [8–11].

The reasoning presented above [6] suggests that *horizonless* spacetimes which possess ergoregions may generally be unstable to superradiant scattering of bosonic fields in their ergoregions. This suggestive argument was raised long ago by Friedman [12]. Most recently, Oliveira et. al. [6] have explored analogous ergoregion instabilities which may develop in fluid flow dynamics. It was first

shown by Unruh [13] that the characteristic wave equation for sound waves propagating inside fluids is analogous to the Klein-Gordon wave equation for massless scalar fields propagating in curved spacetimes [see Eq. (7) below].

Oliveira et. al. [6] have studied numerically the *hydrodynamic vortex* model, a two-dimensional purely circulating flow of a vorticity free ideal fluid. This acoustic system is analogous to an effective horizonless spacetime which nevertheless possesses an ergoregion [14]. In accord with the arguments presented in [12], it has been established in [6] that the effective spacetime geometry which corresponds to the hydrodynamic vortex system is characterized by unstable acoustic perturbation modes. This is an ergoregion instability [6] which is related to the absence of an event horizon in the effective rotating spacetime.

A remarkable feature of the hydrodynamic vortex system is the existence, for each given value of the sound mode harmonic index m [see Eq. (8) below], of *marginally-stable* fluid configurations. These stationary configurations mark the boundary between stable and unstable fluid flows. The main goal of the present study is to obtain an *analytical* formula which describes the physical properties of the hydrodynamic vortex system in its critical (marginally-stable) state.

II. DESCRIPTION OF THE SYSTEM

We study the dynamics of a vorticity free barotropic ideal fluid. Assuming a two-dimensional purely circulating flow in the xy plane, the background (unperturbed) fluid velocity is characterized by [6]

$$v_r = v_z = 0 \quad ; \quad v_\phi = v_\phi(r) . \quad (2)$$

Here r and ϕ are respectively the radial and azimuthal coordinates in the xy plane, and z is the coordinate per-

pendicular to the plane of flow.

Irrotationality of the fluid flow (vorticity free flow) implies that the tangential component of the velocity field, v_ϕ , is given by [6]

$$v_\phi = \frac{C}{r}, \quad (3)$$

where the constant C characterizes the circulation strength of the fluid. Conservation of angular momentum yields

$$\rho v_\phi r = \text{const.} \quad (4)$$

which, together with Eq. (3), implies that the fluid background density ρ is constant. The assumption of a barotropic fluid then implies that the background pressure P and the speed of sound c are also constants.

The two-dimensional circulating fluid flow produces an effective acoustic spacetime, known as the hydrodynamic vortex [6, 14–16], which is characterized by the non-trivial [17] line element

$$ds^2 = -c^2 \left(1 - \frac{C^2}{c^2 r^2}\right) dt^2 + dr^2 - 2C dt d\phi + r^2 d\phi^2 + dz^2. \quad (5)$$

The rotating acoustic spacetime geometry (5) possesses an ergoregion whose outer boundary is determined by the circle at which the fluid flow velocity, $|C|/r$, equals the speed of sound c [6, 14–16]:

$$r_{\text{ergo}} = \frac{|C|}{c}. \quad (6)$$

(We shall henceforth use units in which $c = 1$. Note that in these units C has the dimensions of length).

We shall now consider small perturbations to the background fluid flow. These perturbations (sound waves) propagate in the acoustic spacetime and their linearized Navier-Stokes dynamics is governed by the Klein-Gordon wave equation [6, 13, 18]

$$\nabla^\nu \nabla_\nu \Psi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \Psi \right) = 0. \quad (7)$$

It proves useful to decompose the perturbation field Ψ in the form [19]

$$\Psi(t, r, \phi, z) = \frac{1}{\sqrt{r}} \sum_{m=-\infty}^{\infty} \psi_m(r; \omega) e^{im\phi} e^{-i\omega t}, \quad (8)$$

The ϕ -periodicity of the angular function $e^{im\phi}$ enforces the azimuthal harmonic index $|m|$ to be an integer. Substituting the decomposition (8) into the Klein-Gordon wave equation (7), one obtains the characteristic radial equation

$$\left[\frac{d^2}{dr^2} + \left(\omega - \frac{Cm}{r^2} \right)^2 - \frac{m^2 - \frac{1}{4}}{r^2} \right] \psi_m(r; \omega) = 0 \quad (9)$$

for each field mode [20]. (We shall henceforth omit the harmonic index m for brevity).

III. BOUNDARY CONDITIONS

The background velocity field (3) is singular at the origin, signaling a breakdown of the physical description. In order to mimic a possible experimental scenario in the laboratory, Oliveira et. al. [6] have suggested to impose physically acceptable boundary conditions at a *finite* radial location, $r = r_{\min}$. In particular, it was assumed in [6] that an infinitely long cylinder of radius r_{\min} made of a certain material with acoustic impedance Z [21] is placed at the center of the fluid system.

Oliveira et. al. [6] considered two types of boundary conditions (BCs) at the surface $r = r_{\min}$ of the central cylinder, characterizing two limiting values of the cylinder acoustic impedance: Extremely low- Z materials [21] are characterized by the Dirichlet-type boundary condition [6]:

$$\psi(r = r_{\min}) = 0 \quad , \quad \text{BCI}, \quad (10)$$

whereas extremely high- Z materials [21] (that is, a very rigid boundary cylinder [6]) are characterized by the Neumann-type boundary condition [6]:

$$\frac{d\Psi}{dr}(r = r_{\min}) = \frac{d(\psi/\sqrt{r})}{dr}(r = r_{\min}) = 0 \quad , \quad \text{BCII}. \quad (11)$$

Following [6], we shall consider purely outgoing waves at large distances from the cylinder:

$$\psi(r \rightarrow \infty) \sim e^{i\omega r}. \quad (12)$$

IV. THE ERGOREGION INSTABILITY OF THE HYDRODYNAMIC VORTEX

As emphasized above, the instability of the hydrodynamic vortex system studied in [6] is closely related to the phenomenon of superradiant scattering [14] of sound waves in the ergoregion of the effective spacetime geometry (5). Thus, the simple inequality [6]

$$r_{\min} < C \quad (13)$$

acts as a necessary requirement for the triggering of the ergoregion instability. It simply states that the ergoregion [whose outer boundary is given by $r = r_{\text{ergo}} = C$, see Eq. (6)] must be part of the physical system.

It should be emphasized, however, that not every hydrodynamic vortex system with $r_{\min} < r_{\text{ergo}} = C$ is unstable under perturbations of the m th sound mode [6]. In particular, a remarkable feature of the hydrodynamic vortex system is the existence, for each given set (C, m) of the fluid and field parameters, of a critical cylinder radius,

$$r_{\min} = r_{\min}^*(C, m), \quad (14)$$

which supports stationary (*marginally-stable*) fluid configurations.

The critical (maximal) cylinder radius (14) marks the boundary between stable and unstable composed fluid-cylinder configurations: composed systems whose cylinder radius lies in the regime $r_{\min} > r_{\min}^*(C, m)$ are stable under perturbations of the m th sound mode (that is, the sound mode decays in time), whereas composed systems whose cylinder radius lies in the regime $r_{\min} < r_{\min}^*(C, m)$ are unstable under perturbations of the m th sound mode (that is, the sound mode grows exponentially over time).

V. ONSET OF THE ERGOREGION INSTABILITY IN THE HYDRODYNAMIC VORTEX

The boundary conditions (10)-(12) single out a discrete set of complex resonances $\{\omega_n(C, m, r_{\min})\}$ [22]. These quasinormal resonances characterize the temporal response of the hydrodynamic system to external (sound mode) perturbations. Note, in particular, that stable (exponentially suppressed) sound modes are characterized by $\Im\omega < 0$, whereas unstable (growing in time) sound modes are characterized by $\Im\omega > 0$. The stationary (marginally-stable) resonances, which are the solutions we shall be interested in in this study, are characterized by $\Im\omega = 0$.

In this section we shall analyze the marginally-stable ($\Im\omega = 0$) resonances of the hydrodynamic vortex system. As we shall show below, this hydrodynamic fluid system is actually characterized by genuine *static* resonances with

$$\Re\omega = \Im\omega = 0. \quad (15)$$

In particular, we shall now find the *discrete* set of critical cylinder radii, $\{r_{\min}^*(C, m; n)\}$, which support these marginally-stable fluid configurations.

Remarkably, the radial equation (9) can be solved *analytically* for the marginally-stable modes (15). The general solution of Eq. (9) with $\omega = 0$ can be expressed in terms of the Bessel functions of the first and second kinds (see Eq. 9.1.53 of [23]):

$$\psi(r; \omega = 0) = Ar^{\frac{1}{2}}J_m\left(\frac{Cm}{r}\right) + Br^{\frac{1}{2}}Y_m\left(\frac{Cm}{r}\right), \quad (16)$$

where A and B are normalization constants. The asymptotic large- r limit ($Cm/r \rightarrow 0$) of Eq. (16) is given by (see Eqs. 9.1.7 and 9.1.9 of [23])

$$\begin{aligned} \psi(r \rightarrow \infty; \omega = 0) &= \frac{A}{m!} \left(\frac{Cm}{2}\right)^m r^{-m+\frac{1}{2}} \\ &\quad - \frac{B(m-1)!}{\pi} \left(\frac{Cm}{2}\right)^{-m} r^{m+\frac{1}{2}}. \end{aligned} \quad (17)$$

A physically acceptable (finite) solution at infinity requires $B = 0$, which implies

$$\psi(r; \omega = 0) = Ar^{\frac{1}{2}}J_m\left(\frac{Cm}{r}\right). \quad (18)$$

Taking cognizance of the boundary conditions (10)-(11) which characterize the acoustic properties of the central cylinder, one finds that the marginally-stable resonances (15) of the hydrodynamic vortex system correspond to the following *discrete* radii of the cylinder:

$$r_{\min}^*(C, m; n) = \frac{Cm}{j_{m,n}} \quad , \quad \text{BCI} \quad , \quad (19)$$

and

$$r_{\min}^*(C, m; n) = \frac{Cm}{j'_{m,n}} \quad , \quad \text{BCII} \quad , \quad (20)$$

where $n = 1, 2, 3, \dots$. Here $j_{m,n}$ is the n th positive zero of the Bessel function $J_m(x)$ and $j'_{m,n}$ is the n th positive zero of its first spatial derivative $J'_m(x)$. The real zeros of these functions were studied by many authors, see e.g. [23, 24].

For large values of the acoustic harmonic index, $m \gg n$, one may use the asymptotic relations (see Eqs. 9.5.14 and 9.5.16 of [23]) $j_{m,n} = m[1 + b_n m^{-2/3} + O(m^{-4/3})]$ and $j'_{m,n} = m[1 + b'_n m^{-2/3} + O(m^{-4/3})]$ [25]. Substituting these relations into Eqs. (19) and (20), one finds

$$r_{\min}^*(C, m \gg n; n) = C[1 - b_n m^{-2/3} + O(m^{-4/3})] \quad , \quad \text{BCI} \quad , \quad (21)$$

and

$$r_{\min}^*(C, m \gg n; n) = C[1 - b'_n m^{-2/3} + O(m^{-4/3})] \quad , \quad \text{BCII} \quad . \quad (22)$$

It is worth emphasizing that Eqs. (21)-(22) provides an analytical quantitative explanation for the asymptotic large- m behavior of the hydrodynamic vortex system as numerically presented in Fig. 5 of [6].

For large overtone numbers, $n \gg m$, one may use the asymptotic relations (see Eqs. 9.5.12 and 9.5.13 of [23]) $j_{m,n} = (n + m/2 - 1/4)\pi + O(m^2/n)$ and $j'_{m,n} = (n + m/2 - 3/4)\pi + O(m^2/n)$. Substituting these relations into Eqs. (19) and (20), one finds

$$r_{\min}^*(C, m; n \gg m) = \frac{Cm}{\pi n} [1 + O(m/n)] \quad (23)$$

for both types of boundary conditions. The relation (23) implies that, large overtone modes must be supported by small radii cylinders in order to be able to trigger super-radiant instabilities in the hydrodynamic vortex system.

VI. ANALYTICAL VS. NUMERICAL RESULTS

We shall now compare the predictions of the analytically derived formulas (19)-(20) for the critical cylinder radii, $r_{\min}^*(C, m; n)$, with the corresponding numerical data recently published by Oliveira et. al. [6]. In Table I we present such a comparison, from which one finds a remarkably excellent agreement between the analytical formulas (19)-(20) and the numerical results of [6].

BC	m	r_{\min}^* (Numerical)	r_{\min}^* (Analytical)
I	1	0.13	0.1305
I	2	0.19	0.1947
II	1	0.27	0.2716
II	2	0.33	0.3274

TABLE I: Marginally stable resonances of the hydrodynamic vortex system. We display the critical cylinder radii, r_{\min}^* , for the fundamental $n = 1$ sound mode with fluid circulation $C = 0.5$ as obtained from the analytically derived formulas (19) and (20). We also display the corresponding critical radii as extracted from the numerical data presented in [6]. One finds a remarkably good agreement between the analytical formulas (19)-(20) and the numerically computed [6] critical cylinder radii.

VII. SUMMARY AND DISCUSSION

In this paper, we have used analytical tools in order to analyze the *marginally-stable* resonances of the hydrodynamic vortex system, an effective spacetime geometry for sound waves. These resonances are fundamental to the physics of sound waves in the hydrodynamic acoustic spacetime: in particular, they mark the onset of the superradiant instability in the hydrodynamic vortex system.

In order to mimic a possible experimental scenario in the laboratory, Oliveira et. al. [6] have recently suggested to place a long cylinder of radius r_{\min} at the center of the fluid system, on which physically acceptable

boundary conditions [see Eq. (10) and (11)] are imposed. A remarkable feature of the hydrodynamic vortex system is the existence, for each given set (C, m) of the fluid and sound mode parameters, of a critical cylinder radius, $r_{\min} = r_{\min}^*(C, m)$, which supports marginally-stable fluid configurations.

In the present study we have derived the characteristic resonance conditions [see Eq. (19) and (20)] for these marginally-stable composed fluid-cylinder configurations. In particular, it was shown that the critical cylinder radius r_{\min}^* (the cylinder radius which, for given parameters C and m of the system, marks the boundary between stable and unstable fluid-cylinder configurations) can be expressed in terms of the simple zeros of the Bessel function and its first derivative.

It was shown that the *analytically* derived formulas for the critical cylinder radius [see Eq. (19) and (20)] agree with the recently published *numerical* data for the hydrodynamic vortex system.

Finally, it is worth emphasizing that the physical significance of the critical cylinder radius, r_{\min}^* , lies in the fact that it is the *outermost* location of the cylinder which allows the extraction of rotational energy from the circulating fluid (from the ergoregion of the effective spacetime geometry).

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Yael Oren, Arbel M. Ongo and Ayelet B. Lata for stimulating discussions.

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